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Green functions and the Dirichlet spectrum. (English summary)
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This article has four types of results.

(1) Let $\Omega \subset M$ be a relatively compact open subset with smooth boundary $\partial\Omega \neq \varnothing$ and consider the weighted Laplace operator $\Delta_\mu\colon C_0^\infty(\Omega) \to C_0^\infty(\Omega)$, acting on $C_0^\infty(\Omega)$, the space of smooth functions with compact support on Ω, defined by

$$\Delta_\mu = \frac{1}{\psi}\operatorname{div}(\psi \cdot \operatorname{grad}).$$

The weighted Laplace operator is densely defined and symmetric with respect to $L^2(\Omega,\mu)$-inner product; however, as in the classical case, it is not self-adjoint. Consider the Sobolev spaces $W_0^1(\Omega,\mu)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{W^1(\Omega,\mu)} := \int_\Omega u^d\mu + \int_\Omega |\operatorname{grad} u|^2 d\mu$$

and $W_0^2(\Omega,\mu)$, formed by those functions $u \in W_0^1(\Omega,\mu)$ whose weak Laplacian $\Delta_\mu u$ exists and belongs to $L^2(\Omega,\mu)$, i.e.,

$$W_0^2(\Omega,\mu) = \left\{u \in W_0^1(\Omega,\mu) : \Delta_\mu u \in L^2(\Omega,\mu)\right\}.$$

The operator $\mathcal{L} = -\Delta_\mu|_{W_0^2(\Omega,\mu)}$ is a self-adjoint, non-negative definite elliptic operator and its spectrum, denoted by $\sigma(\Omega,\mu)$, is a discrete increasing sequence of non-negative real numbers $\{\lambda_k\}_{k=1}^\infty \subset [0,\infty)$ (counted according to multiplicity), with $\lim_{k\to\infty}\lambda_k = \infty$.

The weighted Green operator $G^\Omega\colon L^2(\Omega,\mu) \to L^2(\Omega,\mu)$, given by

$$G^\Omega(f)(x) = \int_0^\infty \int_\Omega p_t^\Omega(x,y) f(y) d_\mu(y) dt,$$

where $p_t^\Omega(x,y)$ is the heat kernel of the operator $\mathcal{L} = -\Delta_\mu|_{W_0^2(\Omega,\mu)}$ is a bounded self-adjoint operator, inverse to $\mathcal{L}$, i.e., $G^\Omega = \mathcal{L}^{-1}$. For any $f \in L^2(\Omega,\mu)$ there is a unique solution $u = G^\Omega(f) \in W_0^2(\Omega,\mu)$ to the equation $-\Delta_\mu u = f$. In particular, by applying G^Ω to the ith-eigenvalue equation

$$\Delta_\mu u_i + \lambda_i(\Omega) u_i = 0,$$

the ith-eigenvalue $\lambda_i(\Omega) = u_i/G^\Omega(u_i)$ is obtained. The difficulty with this approach, in order to obtain the ith-eigenvalue, is that one needs to know the Green operator and the ith-eigenfunction beforehand. To overcome this inconvenience, a bootstrapping argument due to S. Sato [Math. Z. **181** (1982), no. 3, 313–318; MR0678887] is applied to show that the first eigenvalue $\lambda_1(\Omega)$ can be obtained as

$$\lambda_1(\Omega) = \lim_{k\to\infty} \frac{\left\|G^k(f)\right\|_{L_2}}{\left\|G^{k+1}(f)\right\|_{L_2}}$$

for any positive function $f \in L^2(\Omega,\mu)$ where $\|u\|_{L^2} = \sqrt{\int_\Omega |u|^2 d\mu}$ is the $L^2(\Omega,\mu)$-norm and $G^k = \overbrace{G \circ \cdots \circ G}^{k \text{ times}}$.

(2) The set $\{\mathcal{A}_k\}_{k=1}^{\infty}$, $\mathcal{A}_k = \int_\Omega \phi_k d\mu$ is called the $L^1(\Omega, \mu)$-moment spectrum of Ω. The $L^1(\Omega, \mu)$-moment spectrum is intertwined with the exit time and the Dirichlet spectrum of Ω. Consider the hierarchy Dirichlet problem

$$\begin{cases} \phi_0 = 1 & \text{in } \Omega, \\ \Delta_\mu \phi_k + k\phi_{k-1} = 0 & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for the Laplacian on a bounded open subset with smooth boundary $\Omega \subset M$. Suppose X_t is a Brownian motion in Ω and $\tau = \inf\{t \geq 0 : X_t \notin \Omega\}$ is the first exit time from Ω. If ϕ_k is the solution of (1) then $\mathbb{E}^x[\tau^k] = \phi_k(x)$. The generator of the process X_t associated to weighted Laplacian preserves the relationship between the exit time and the solutions of the hierarchy problem (1) as in the Laplacian case. In Theorem 3.2 of the present paper it is shown that $\phi_k = k! G^k(1)$, where G is the Δ_μ-Green operator of Ω. This formula was proven by P. T. McDonald and R. Meyers, in [J. Funct. Anal. **200** (2003), no. 1, 150–159 (Formula 2.9, Proposition 2.2); MR1974092], in the non-weighted setting. The work of A. Hurtado, S. Markvorsen and V. Palmer Andreu [Math. Ann. **365** (2016), no. 3-4, 1603–1632; MR3521100] is extended to general bounded domains on the first eigenvalue of rotationally invariant balls $B_h(o, r)$ by proving $\lambda_1(\Omega) = \lim_{k\to\infty} k\mathcal{A}_{k-1}/\mathcal{A}_k$.

(3) The spectrum of $\mathcal{L} = -\Delta_\mu|_{W_0^2(\Omega,\mu)}$, sometimes called the spectrum of Ω and denoted by $\sigma(\Omega)$, is the set of all $\lambda \in [0, \infty)$ for which $\mathcal{L} - \lambda I$ is not injective or the inverse operator $(\mathcal{L} - \lambda I)^{-1}$ is unbounded. The set of all λ for which $(\mathcal{L} - \lambda I)$ is not injective is called the *point spectrum* and denoted by $\sigma_p(\Omega)$. The elements of $\sigma_p(\Omega)$ are the eigenvalues of $\mathcal{L}$. Each eigenvalue $\lambda \in \sigma_p(\Omega)$ defines an associated vector space $V_\lambda = \{u \in L^2(\Omega) : \Delta u + \lambda u = 0\}$. The set of all isolated eigenvalues of finite multiplicity, i.e., those $\lambda \in \sigma_p(\Omega)$ for which there exists $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon) \cap \sigma(\Omega) = \{\lambda\}$ and $\dim(V_\lambda) < \infty$, is called the *discrete spectrum* and it is denoted by $\sigma_d(\Omega)$. The complement of the discrete spectrum is the *essential spectrum*, $\sigma_{\text{ess}}(\Omega) = \sigma(M)/\sigma_d(\Omega)$. When Ω is a bounded open set with smooth boundary $\partial\Omega$ (possibly empty), then the spectrum of $\mathcal{L}$ is discrete, i.e., a sequence of non-negative real numbers

$$0 \leq \overline{\lambda}_1(\Omega) < \overline{\lambda}_2(\Omega) < \cdots \nearrow \infty,$$

where each associated eigenspace $V_k = \{\phi : \Delta\phi + \overline{\lambda}_k + \phi = 0\}$ is finite-dimensional and $L^2(\Omega) = \bigoplus_{k=1}^{\infty} V_k$. Moreover, $\phi \in V_k \Rightarrow \phi \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and $\phi|\partial\sigma = 0$. In this paper, the radial spectrum $\sigma^{\text{rad}}(B_h(o, r))$ to an isoperimetric quotient is studied.

(4) A proper minimal surface $M \subset \mathbb{R}^3$ and the extrinsic ball $\Omega = M \cap B_{\mathbb{R}^3}(o, r)$ are considered. Upper and lower estimates for the series $\sum \lambda_i^{-2}(\Omega)$ are obtained in terms of the volume $\operatorname{vol}(\Omega)$ and the radius r of the extrinsic ball Ω. *F. Ayca Cetinkaya*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.