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Guliyev, Namig J. (AZ-AOS)

Essentially isospectral transformations and their applications. (English summary)

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The properties of Darboux-type transformations of Sturm-Liouville problems for the equations

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, \pi),$$

with boundary conditions

$$\frac{y'(0)}{y(0)} = -f(\lambda), \quad \frac{y'(\pi)}{y(\pi)} = F(\lambda)$$

are studied, where $q \in \mathcal{L}_1(0, \pi)$ is a real-valued function,

$$f(\lambda) = h_0\lambda + h + \sum_{k=1}^d \frac{\delta_k}{h_k - \lambda} \quad \text{and} \quad F(\lambda) = H_0\lambda + H + \sum_{k=1}^D \frac{\Delta_k}{H_k - \lambda}$$

are rational Herglotz-Nevanlinna functions with real coefficients, i.e., $h_0, H_0 \geq 0$, $\delta_k, \Delta_k > 0$, $h_1 < \dots < h_d$ and $H_1 < \dots < H_D$.

Using these transformations, various direct and inverse spectral results for these problems are obtained and presented in a unified manner, among which are asymptotics of eigenvalues and norming constants, oscillation of eigenfunctions, regularized trace formulas and inverse uniqueness and existence theorems. *F. Ayca Cetinkaya*

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