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Recovering differential operators with nonlocal boundary conditions. (English summary)
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In this paper the authors consider the differential equation

$$-y''(x)+q(x)y(x)=\lambda y(x),\quad x\in(0,T), \tag{1}$$

and two linear forms

$$U_j(y):=\int_0^T y(t)d\sigma_j(t),\quad j=1,2,$$

where $q\in L(0,T)$ is a complex-valued function, and the $\sigma_j(t)$ $(j=1,2)$ are complex-valued functions of bounded variation which are continuous from the right for $t>0$ and such that the limits $H_j:=\sigma_j(+0)-\sigma_j(0)$ are finite. Throughout the paper the authors assume $H_1\neq 0$ and they mention a natural assumption that $U_1\neq kU_2$ holds for all $k\in\mathbf{C}$, that is, two linear forms U_j $(j=1,2)$ are linearly independent.

In the paper, the functions $X_k(x,\lambda)$ and $Z_k(x,\lambda)$ $(k=1,2)$ are defined as the solutions of equation (1) under the initial conditions

$$X_1(0,\lambda)=X_2'(0,\lambda)=Z_1(T,\lambda)=Z_2'(T,\lambda)=1,$$
$$X_1'(0,\lambda)=X_2(0,\lambda)=Z_1'(T,\lambda)=Z_2(T,\lambda)=0.$$

The boundary value problem (BVP) L_0 for equation (1) with the condition

$$U_1(y)=U_2(y)=0$$

is considered and it is assumed that $w(\lambda):=\det[U_j(X_k)]_{j,k=1,2}\neq 0$. It is stated that $\lambda\mapsto w(\lambda)$ is an entire function of order 1/2 and its zeros (counting multiplicity) coincide with the eigenvalues of L_0. The function $w(\lambda)$ is called the characteristic function for L_0.

The boundary value problem L_j $(j=1,2)$ for equation (1) is determined by the condition $U_j(y)=V_1(y)=0$, where $V_j:=y^{(j-1)}(T)$ $(j=1,2)$, and it is mentioned that the eigenvalue set $\Lambda_j=\{\lambda_{nj}\}_{n\geq 1}$ (counting multiplicity) of the BVP L_j coincides with the zeros of the characteristic function $\Delta_j(\lambda):=\det[U_j(X_k),V_1(X_k)]_{k=1,2}$.

For $\lambda\neq\lambda_{n1}$, let $\Phi(x,\lambda)$ be the solution of equation (1) under the conditions $U_1(\Phi)=1$, $V_1(\Phi)=0$. Let $M(\lambda):=U_2(\Phi)=1$. The function $M(\lambda)$ is called a Weyl-type function. It is known from [G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville problems and their applications*, Nova Sci. Publ., Huntington, NY, 2001; MR2094651] that for Sturm-Liouville operators with classical two-point separated boundary conditions, the specification of the Weyl function uniquely determines the potential $q(x)$. But in the case considered here, involving nonlocal boundary conditions, the authors argue that this is not true: the specification of the Weyl-type function $M(\lambda)$ does not uniquely determine the potential (some counterexamples are shown in Section 4). Thus the inverse problem is formulated as follows:

Inverse Problem I. Given $M(\lambda)$ and $w(\lambda)$, construct the potential $q(x)$.

In the paper the BVP L_{11} for equation (1) with the condition $U_1(y)=V_2(y)=0$ is considered. The eigenvalue set $\Lambda_{11}:=\{\lambda_n^1\}_{n\geq 1}$ of the BVP L_{11} coincides with the zeros of the characteristic function $\Delta_{11}(\lambda) := \det[U_1(X_k), V_2(X_k)]$. The corresponding inverse problem is stated as follows:

Inverse Problem II. Given $\{\lambda_n^1, \lambda_{n1}^1\}_{n\geq 1}$, construct $q(x)$.

The authors mention that this inverse problem is a generalization of the well-known Borg inverse problem [G. Borg, Acta Math. **78** (1946), 1–96; MR0015185] for Sturm-Liouville operators with classical two-point separated boundary conditions, and coincides with it when $U_1(y) = y(0)$.

The paper consists of six sections. In Section 1, the authors suggest statements of the inverse problems and formulate the main results. Section 2 introduces important notions and properties of spectral characteristics. The proofs of the main theorems are given in Section 3. Section 4 is devoted to presenting counterexamples related to the statements of the inverse problems. Additional spectral data are introduced in Section 5. In Section 6 an inverse problem of recovering the potential q from the given three spectra is considered as an example. *F. Ayca Cetinkaya*

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