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A practical method for recovering Sturm-Liouville problems from the Weyl function. (English summary)

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In this paper, the authors consider the inverse problem of recovering the potential of a Sturm-Liouville equation on a finite interval together with the boundary conditions from values of the Weyl-Titchmarsh m-function on a given set of points.

For some inverse spectral problems the given spectral data allow one to obtain the values of the m-function at any point $z \in \mathbf{C}$, while for many problems only the values of the m-function on a countable set of points $\{z_k\}_{k=0}^{\infty}$ can be obtained. Thus one faces two principal questions. Is it possible to recover uniquely the m-function from the given values $m(z_k)$, $k = 0, 1, \ldots$? If yes, how can this be done?

Some computational analyses for finding the m-function were given in [R. R. del Río Castillo, F. Gesztesy and B. Simon, Internat. Math. Res. Notices **1997**, no. 15, 751–758; MR1470376] and [M. Horváth, Ann. of Math. (2) **162** (2005), no. 2, 885–918; MR2183284]. The proofs in these analyses often lead to elegant mathematics; however, they are not useful as far as constructive methods are concerned.

The only constructive method for finding the m-function was given in [N. P. Bondarenko, Open Math. **18** (2020), no. 1, 512–528; MR4113762; C.-F. Yang, N. P. Bondarenko and X.-C. Xu, Inverse Probl. Imaging **14** (2020), no. 1, 153–169; MR4043129] (based on the ideas of W. Rundell and P. E. Sacks [Math. Comp. **58** (1992), no. 197, 161–183; MR1106979] and K. Chadan et al. [*An introduction to inverse scattering and inverse spectral problems*, SIAM Monogr. Math. Model. Comput., SIAM, Philadelphia, PA, 1997 (Chapter 3); MR1445771]). This algorithm recovers a certain combination of the integral kernel of the transmutation and its derivative by using non-harmonic Fourier series. However, it works only for sequences z_n such that the sequence of functions $\{e^{\pm i\sqrt{z_n}t}\}_{n\in\mathbf{N}}$ is a Riesz basis in $L_2(-2\pi, 2\pi)$ and requires a beforehand knowledge of the parameter ω (depending on the average of the potential q and boundary parameter H); moreover, a numerical realization of this method is expected to converge slowly (no numerical illustration was provided in [N. P. Bondarenko, op. cit.] or [C.-F. Yang, N. P. Bondarenko and X.-C. Xu, op. cit.]).

In the present paper, the authors use Fourier-Legendre series to represent the transmutation integral kernel and its derivative. This idea was originally proposed in [V. V. Kravchenko, L. J. Navarro Méndez and S. M. Torba, Appl. Math. Comput. **314** (2017), 173–192; MR3683865] and resulted in a Neumann series Bessel-function (NSBF) representation of the solutions as well as their derivatives of Sturm-Liouville equations. As a result, a constructive algorithm for recovering the m-function from its values on a countable set of points is proposed. It reduces to an infinite system of linear equations, and the unique solvability of this system is proved.

The proposed method leads to an efficient numerical algorithm. Since the NSBF representation possesses the remarkable property of a uniform error bound for all real $\lambda > 0$, one can obtain any finite set of approximate eigenvalues and corresponding norming constants with a non-deteriorating accuracy. That is, an efficient conversion of the original problem to an inverse problem with a given spectral density function is proposed. The method of solution introduced in [V. V. Kravchenko, J. Inverse

Ill-Posed Probl. **27** (2019), no. 3, 401–407; MR3962689; *Direct and inverse Sturm-Liouville problems: a method of solution*, Front. Math., Birkhäuser/Springer, Cham, 2020, doi:10.1007/978-3-030-47849-0] and refined in [V. V. Kravchenko and S. M. Torba, Inverse Problems **37** (2021), no. 1, Paper No. 015015; MR4197838] is adapted to recover the potential and the boundary conditions from the spectral density function.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.