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★**On accelerants and their analogs, and on the characterization of the rectangular Weyl functions for Dirac systems with locally square-integrable potentials on a semi-axis. (English summary)**

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In this paper, the author considers the self-adjoint Dirac system

$$\frac{d}{dx}y(x,z) = i(zj + jV(x))y(x,z), \quad x \geq 0, \tag{1}$$

where

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix},$$

in which $m_1 + m_2 =: m$, I_{m_k} is the $m_k \times m_k$ identity matrix and $v(x)$ is an $m_1 \times m_2$ matrix function.

The paper is a continuation of [A. L. Sakhnovich, J. Spectr. Theory **5** (2015), no. 3, 547–569; MR3416832], where an inverse problem for Dirac systems with locally square-integrable potentials on a semi-axis was solved, and a procedure to recover the $m_1 \times m_2$ potential v from the Weyl function was given. In the present paper, the author characterizes the Weyl matrix functions corresponding to such Dirac systems and an alternative method to recover v from the Weyl function is demonstrated.

The study of the inverse spectral problem for Dirac systems goes back to the seminal paper [Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 637–640; MR0080735] by M. G. Kreĭn, where the case of a continuous scalar potential v was dealt with. An important feature of Kreĭn's approach is establishing a deep connection between Dirac systems and convolution operators, which play a crucial role in solving the inverse problem. The kernel of the corresponding positive convolution (integral) operator is called the accelerant. In the case of the Dirac system (1) with rectangular matrix functions v, one cannot use convolution operators anymore. However, structured operators of the form (3) (see below) of the paper which satisfy operator identities (4) of the paper, similar to those for convolution operators, are applied for solving inverse problems [B. Fritzsche et al., Inverse Problems **28** (2012), no. 1, 015010; MR2864505; A. L. Sakhnovich, op. cit.; A. L. Sakhnovich, L. A. Sakhnovich and I. Y. Roitberg, *Inverse problems and nonlinear evolution equations*, De Gruyter Stud. Math., 47, De Gruyter, Berlin, 2013; MR3098432]. In particular, the matrix function Φ_1' in (3) (or, equivalently in (5)) may be considered as a direct analog of Kreĭn's accelerant.

In the paper, it is assumed that the potential v of the Dirac system (1) is locally square-integrable. Let $u(x,z)$ stand for the fundamental solution of (1) normalized by the condition

$$u(0,z) = I_m.$$

Definition 1. A Weyl-Titchmarsh (or simply Weyl) function of the Dirac system (1) on $[0,\infty)$, where the potential v is locally integrable, is an $m_2 \times m_1$ matrix function φ

which satisfies the inequality

$$\int_0^\infty [\, I_{m_1} \quad 0\,]\, u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}^+;$$

here $\mathbb{C}_+$ stands for the open upper half-plane of the complex plane.

Remark 1. In [A. L. Sakhnovich, op. cit.] it was shown that the Weyl function $\varphi(z)$ always exists in $\mathbb{C}_+$ and that it is unique, holomorphic and contractive (i.e., $\varphi(z)^*\varphi(z) \le I_{m_1}$).

Since φ is contractive, the function below is well defined:

$$\Phi\left(\frac{x}{2}\right) = \frac{1}{\pi} e^{x\eta}\, \text{l.i.m.}_{a\to\infty} \int_{-a}^{a} e^{-ix\xi} \frac{\xi + i\eta}{2i(\xi+i\eta)} d\xi, \quad \eta > 0. \tag{2}$$

Here, l.i.m. stands for the entrywise limit in the norm of $L^2(0,b)$, $0 < b \le \infty$. Note that if we put additionally $\Phi_1(x)$ for $x < 0$, equality (2) holds for l.i.m. as the entrywise limit in $L^2(-b,b)$. The matrix function Φ_1 does not depend on $\eta > 0$. Moreover, the following statement is valid.

Proposition 1. Let φ be the Weyl function of the Dirac system (1) on $[0,\infty)$, where the potential v is locally square-integrable. Then Φ_1 given on $\mathbb{R}$ by (2) is absolutely continuous, $\Phi_1(x) \equiv 0$ for $x \le 0$, Φ_1' is locally square-integrable on $\mathbb{R}$, and the operators

$$S_\xi = I - \frac{1}{2}\int_0^\xi \int_{|x-t|}^{x+t} \Phi_1'\left(\frac{\xi + x - t}{2}\right) \Phi_1'\left(\frac{\xi + t - x}{2}\right)^* d\xi dt \tag{3}$$

are positive definite and boundedly invertible in $L^2(0,\xi)$ $(0 < \xi < \infty)$.

Here (as usual), $\Phi_1' := \frac{d}{dx}\Phi$. The result above (on the properties of the Weyl functions) solves the direct spectral problem. Below, we formulate Theorem 1, which characterizes the set of Weyl functions.

Remark 2. The operator S_ξ given by (3) (with an absolutely continuous $m_2 \times m_1$ matrix function $\Phi_1(x)$ such that $\Phi_1(0) = 0$ and $\Phi_1'(x)$ is square-integrable on $[0,\xi]$) is the unique solution of the operator identity

$$A_\xi S_\xi - S_\xi A_\xi^* = i\Pi_\xi j \Pi_\xi^*, \tag{4}$$

where A_ξ is an integration operator in $L^2(0,\xi)$ multiplied by $-i$ and Π_ξ is a multiplication operator ($\Pi_\xi \in B(\mathbb{C}^m, L^2_{m_2}(0,\xi))$). That is, A_ξ and Π_ξ are given by the relations

$$A_\xi = -i\int_0^x dt, \quad \Pi_\xi g = [\, \Phi_1(x) \quad I_{m_1}\,]\, g \quad (g \in \mathbb{C}^m),$$

and Π_ξ is a bounded mapping into $L^2_{m_2}(0,\xi)$.

Changing variables, the author rewrites S_ξ in an equivalent and more convenient form as

$$S_\xi = I - \int_0^\xi s(x,t)dt, \quad s(x,t) := \int_0^{\min(x,t)} \Phi_1'(x-r)\Phi_1'(t-r)^* dr. \tag{5}$$

In order to complete a characterization of Weyl functions, it is assumed that some $m_2 \times m_1$ matrix function has the properties of the Weyl function described in Remark 1 and Proposition 1. The following theorem is the main result of the paper.

Theorem 1. Let an $m_2 \times m_1$ matrix function $\varphi(z)$ be holomorphic and contractive in $\mathbb{C}_+$. Let $\Phi_1(x)$ given by (2) be absolutely continuous, let $\Phi_1(0) = 0$, and let Φ_1' be square-integrable on all the finite intervals $[0,\xi]$. Assume that the operators S_ξ, which are expressed via Φ_1 in (5), are positive definite and boundedly invertible in $L^2(0,\xi)$ $(0 < \xi < \infty)$.

Then φ is the Weyl function of some Dirac system (1) on $[0,\infty)$ such that the potential

v of this Dirac system is locally square-integrable.

{For the collection containing this paper see MR3752616}

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