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Kritskov, L. V. [Kritskov, Leonid V.] (RS-MOSC-NDM)
Classes of uniform convergence of spectral expansions for the one-dimensional Schrödinger operator with a distribution potential. (English summary)
Translation of Differ. Uravn. **53** (2017), no. 5, 591–602.
Differ. Equ. **53** (2017), *no. 5,* 583–594.

In the paper under review, the author assumes that $\mathcal{L}$ is the self-adjoint operator on $L_2(\mathbf{R})$ corresponding to the differential operation

$$lu = -u'' + q(x)u, \quad x \in \mathbf{R}, \tag{1}$$

where the real potential $q(x)$ when restricted to any compact set in $\mathbf{R}$ belongs to the space W_2^{-1}.

According to [R. O. Hryniv and Y. V. Mykytyuk, Methods Funct. Anal. Topology **7** (2001), no. 4, 31–42; MR1879483; Methods Funct. Anal. Topology **18** (2012), no. 2, 152–159; MR2978191] the operator $\mathcal{L}$ exists if the function $q(x)$ belongs to the space W_2^{-1} uniformly locally, or, which is the same, admits the representation

$$q(x) = Q'(x) + q_1(x), \tag{2}$$

in the sense of distributions, where $Q \in L_{2,\mathrm{unif}}(\mathbf{R})$ and $q_1 \in L_{1,\mathrm{unif}}(\mathbf{R})$, i.e.,

$$\begin{aligned} \|Q\|_{2*}^2 &:= \sup_{z\in\mathbf{R}} \int_{|z-y|\le 1} |Q(y)|^2\,dy < \infty, \\ \|q_1\|_{1*} &:= \sup_{z\in\mathbf{R}} \int_{|z-y|\le 1} |q_1(y)|\,dy < \infty. \end{aligned} \tag{3}$$

In this case, the operator $\mathcal{L}$ is semibounded below [see op. cit.; MR1879483]; therefore, adding a constant to the function $q_1(x)$ in the representation (2) is necessary. It is also assumed that the operator $\mathcal{L}$ is nonnegative.

Let $\{E_\lambda, \lambda \ge 0\}$ be the resolution of the identity for the operator $\mathcal{L}$. Then it follows from the spectral theorem for self-adjoint operators in Hilbert spaces that the spectral expansion $E_\lambda f(x)$ of an arbitrary function $f \in L_2(\mathbf{R})$ converges as $\lambda \to +\infty$ to the function $f(x)$ in the metric of $L_2(\mathbf{R})$. To ensure the uniform convergence of the $E_\lambda f(x)$, it is natural to restrict the function class by requiring additional smoothness for the case in which the potential $q(x)$ is uniformly bounded and the function to be expanded belongs to the Sobolev-Liouville class $L_2^\alpha(\mathbf{R})$ with differentiability order $\alpha > \frac{1}{2}$. An estimate in [V. A. Il′in and I. E. Antoniou, Differ. Uravn. **31** (1995), no. 10, 1649–1657, 1773 (1996); MR1433882] was proved in [S. A. Denisov, Differ. Uravn. **33** (1997), no. 6, 754–761, 861; MR1615055] for a wide class of singular differential operations (1) with potential in the class $L_{1,\mathrm{unif}}(\mathbf{R})$. It was shown that the uniform convergence of the spectral expansion $E_\lambda f(x)$ on the entire line $\mathbf{R}$ also holds for functions $f \in L_2^\alpha(\mathbf{R})$, $\alpha > \frac{1}{2}$, and the corresponding estimate of the convergence rate

$$\sup_{x\in\mathbf{R}} |E_\lambda f(x) - f(x)| = o(\lambda^{(1-2\alpha)/4}) \tag{4}$$

is true for $\alpha \in (1/2, 3/2)$. (For $\alpha \ge 3/2$, the estimate (4) remains valid with right-hand side replaced by $o(\lambda^{-1/2-\epsilon})$ with arbitrary $\epsilon > 0$.)

For the positive self-adjoint operator $\mathcal{L}$ corresponding to the operation (1) on the entire line $\mathbf{R}$ with a potential of the form (2), (3), a uniform estimate on the entire line for the increment of the spectral function (that is, the kernel $\theta(x,y;\lambda)$ in the integral representation of $E_\lambda f(x)$) was obtained in [L. V. Kritskov, Differ. Equ. **53** (2017), no. 2, 180–191; MR3636799]. This estimate, together with a standard argument, implies the uniform convergence of $E_\lambda f(x)$ on $\mathbf{R}$ for functions $f(x)$ lying in the domain $D(\mathcal{L}^\beta)$ of the βth power of $\mathcal{L}$ with $\beta > 1/4$.

In the present paper, the author constructs classes of functions $f(x)$ whose spectral expansions $E_\lambda f(x)$ uniformly converge on the entire line $\mathbf{R}$ and the convergence rate is the same as in the estimate (4). The following assertion is the main result of the present paper.

Theorem 1. Let $\mathcal{L}$ be the self-adjoint operator corresponding to the operation (1) with potential $q(x)$ uniformly locally belonging to W_2^{-1}. If the function $f(x)$ belongs to the class $L^\alpha_{2,\epsilon}(\mathbf{R})$, i.e. can be written as

$$f(x) = \int_{\mathbb{R}} G_\lambda(\mu|x-y|)e(x,y)h(y)dy,$$

with exponent $\alpha \in (1/2, 3/2)$ (for details regarding G_λ and e see the paper), then its spectral expansion $E_\lambda f(x)$ converges uniformly and the following estimate holds:

$$\sup_{x\in\mathbf{R}} |E_\lambda f(x) - f(x)| = o(\lambda^{(1-2\alpha)/4}) \, \|h\|_2 \, .$$

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.